

# Widths of embeddings of 2-microlocal Besov spaces <sup>\*</sup>

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## Abstract

We consider the asymptotic behaviour of the approximation, Gelfand and Kolmogorov numbers of compact embeddings between 2-microlocal Besov spaces with weights defined in terms of the distance to a  $d$ -set  $U \subset \mathbb{R}^n$ . The sharp estimates are shown in most cases, where the quasi-Banach setting is included.

**Key words:** Approximation numbers; Gelfand numbers; Kolmogorov numbers; Compact embeddings; 2-microlocal Besov spaces.

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## 1 Introduction

In this paper we investigate the embeddings between 2-microlocal Besov spaces with one special type of weights from the standpoint of certain approximation quantities. More precisely, we are interested in asymptotic behaviour of the approximation, Gelfand and Kolmogorov numbers. This problem has recently been suggested only for entropy numbers by Leopold and Skrzypczak [17]. First, we recall some definitions.

Let  $\varphi$  be a positive function from the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  of infinitely differentiable and rapidly decreasing functions with

$$\varphi(x) = 1 \text{ for } |x| \leq 1 \text{ and } \text{supp } \varphi \subset \{x : |x| \leq 2\}. \quad (1.1)$$

We set  $\varphi_0 = \varphi$  and  $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$  for  $j \in \mathbb{N}$  and  $x \in \mathbb{R}^n$ . This leads to the smooth dyadic resolution  $\{\varphi_j\}_{j \in \mathbb{N}_0}$  of unity, i.e.,  $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ ,  $x \in \mathbb{R}^n$ , so

$$f = \varphi_0(D)f + \sum_{j=1}^{\infty} \varphi_j(D)f, \quad f \in \mathcal{S}'(\mathbb{R}^n),$$

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where

$$\varphi_j(D)f(x) := (2\pi)^{-n} \iint e^{i\xi(x-y)} \varphi_j(\xi) f(y) dy d\xi.$$

For a bounded subset  $U \subset \mathbb{R}^n$ , we denote  $\text{dist}(x, U) = \inf_{y \in U} |x - y|$ , and we define for  $s' \in \mathbb{R}$  the 2-microlocal weights by

$$w_j(x) := (1 + 2^j \text{dist}(x, U))^{s'}, \quad j \in \mathbb{N}_0, \quad x \in \mathbb{R}^n. \quad (1.2)$$

This type of weight sequences is just a typical example for admissible weight sequences. We refer to [10, 12] for detailed discussions of a large class of admissible weight sequences. The case of single weights seems more familiar to us, cf., e.g., [7, 16, 30].

Given  $0 < p, q \leq \infty$  and  $s, s' \in \mathbb{R}$ , we define the 2-microlocal spaces  $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$  by

$$B_{p,q}^{s,s'}(\mathbb{R}^n, U) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{p,q}^{s,s'}(\mathbb{R}^n, U)} < \infty \right\},$$

where

$$\|f\|_{B_{p,q}^{s,s'}(\mathbb{R}^n, U)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \|w_j \varphi_j(D)f\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q}.$$

There is an analogous definition for  $F_{p,q}^{s,s'}(\mathbb{R}^n, U)$ . Moritoh and Yamada introduced in [20] the spaces  $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$  of homogeneous type in case when  $U \subset \mathbb{R}^n$  is open.

2-Microlocal Besov spaces  $B_{p,q}^{s,mloc}(\mathbb{R}^n, w)$  with more general admissible weights were introduced by Kempka [10, 13], and generalized the 2-microlocal spaces  $C_{x_0}^{s,s'}(\mathbb{R}^n)$  introduced by Bony [1] and Jaffard [8] in two directions. We refer to [10, 11, 12, 19] for systematic discussions of this concept, its history and further references.

Following Leopold and Skrzypczak [17], we concentrate on the embeddings,

$$B_{p_1,q_1}^{s_1,s'_1}(\mathbb{R}^n, U) \hookrightarrow B_{p_2,q_2}^{s_2,s'_2}(\mathbb{R}^n, U), \quad (1.3)$$

where  $U$  is a  $d$ -set (the precise definition of  $d$ -sets will be given in Section 3.1).

Our main intention in this paper is to find the optimal asymptotic order of the approximation, Gelfand and Kolmogorov numbers of the embeddings (1.3). Our approach is essentially a combination of [17] and [24] with its corrigendum [25]. In particular, Leopold and Skrzypczak [17] gave a necessary and sufficient condition on the parameters and weights of the 2-microlocal Besov spaces which guarantees compactness of the embeddings (1.3), and determined the entropy estimates for such embeddings. Moreover, our main tools are the use of operator ideals, see [2, 21, 22], and the basic estimates of related widths of the Euclidean ball due to Kashin [9], Gluskin [6] and Edmunds and Triebel [3] with [4, 5, 18, 29].

The paper is structured as follows. In Section 2, we introduce approximation, Gelfand and Kolmogorov numbers, and present our main results. Section 3 represents the most dominant part of this paper; here we adopt a wavelet description of the 2-microlocal Besov spaces  $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$ , and prove their width estimates of embeddings of related sequence spaces. Finally, in Section 4, these results will be used to derive the desired estimates for the function space embeddings under consideration.

Throughout the paper (unless additional restrictions are mentioned) we suppose that

$$s, s_1, s_2, s', s'_1, s'_2 \in \mathbb{R}, \quad 0 < p, p_1, p_2, q, q_1, q_2 \leq \infty, \quad \delta = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0. \quad (1.4)$$

For a real number  $a$ , we define  $a_+ = \max(a, 0)$ . And let  $\frac{1}{p^*} = (\frac{1}{p_2} - \frac{1}{p_1})_+$ .

**Notation 1.1.** By the symbol ' $\hookrightarrow$ ' we denote continuous embeddings.

Identity operators will always be denoted by  $\text{id}$ . Sometimes we do not indicate the spaces where  $\text{id}$  is considered, and likewise for other operators.

Let  $X$  and  $Y$  be complex quasi-Banach spaces and denote by  $\mathcal{L}(X, Y)$  the class of all linear continuous operators  $T : X \rightarrow Y$ . If no ambiguity arises, we write  $\|T\|$  instead of the more exact versions  $\|T|_{\mathcal{L}(X, Y)}\|$  or  $\|T : X \rightarrow Y\|$ .

The symbol  $a_k \preceq b_k$  means that there exists a constant  $c > 0$  such that  $a_k \leq cb_k$  for all  $k \in \mathbb{N}$ . And  $a_k \succeq b_k$  stands for  $b_k \preceq a_k$ , while  $a_k \sim b_k$  denotes  $a_k \preceq b_k \preceq a_k$ .

All unimportant constants will be denoted by  $c$  or  $C$ , sometimes with additional indices.

## 2 Main results

We recall the definitions of the approximation, Gelfand and Kolmogorov numbers, see [21, 23]. We use the symbol  $A \subset\subset B$  if  $A$  is a closed subspace of a topological vector space  $B$ .

**Definition 2.1.** Let  $T \in \mathcal{L}(X, Y)$ .

(i) The  $k$ th approximation number of  $T$  is defined by

$$a_k(T, X, Y) = \inf\{\|T - A\| : A \in \mathcal{L}(X, Y) \text{ with } \text{rank}(A) < k\}, \quad k \in \mathbb{N},$$

also written by  $a_k(T)$  if no confusion is possible. Here  $\text{rank}(A)$  is the dimension of the range of the operator  $A$ .

(ii) The  $k$ th Kolmogorov number of  $T$  is defined by

$$d_k(T, X, Y) = \inf\{\|Q_N^Y T\| : N \subset\subset Y, \dim(N) < k\},$$

also written by  $d_k(T)$  if no confusion is possible. Here,  $Q_N^Y$  stands for the natural surjection of  $Y$  onto the quotient space  $Y/N$ .

(iii) The  $k$ th Gelfand number of  $T$  is defined by

$$c_k(T, X, Y) = \inf\{\|T J_M^X\| : M \subset\subset X, \text{codim}(M) < k\},$$

also written by  $c_k(T)$  if no confusion is possible. Here,  $J_M^X$  stands for the natural injection of  $M$  into  $X$ .

Note that the  $k$ -th approximation, Kolmogorov and Gelfand number are identical to the  $(k - 1)$ -th linear, Kolmogorov and Gelfand width of  $T$ , respectively, see Pinkus [23].

It is well-known that the operator  $T$  is compact if and only if  $\lim_k d_k(T) = 0$  or equivalently  $\lim_k c_k(T) = 0$ , but if  $\lim_k a_k(T) = 0$ , see [23]. The opposite implication for  $a_k(T)$  is not true in general.

Both concepts, Kolmogorov and Gelfand numbers, are related to each other. Namely they are dual to each other in the following sense, cf. [21, 23]: If  $X$  and  $Y$  are Banach spaces, then

$$c_k(T^*) = d_k(T) \quad (2.1)$$

for all compact operators  $T \in \mathcal{L}(X, Y)$  and

$$d_k(T^*) = c_k(T) \quad (2.2)$$

for all  $T \in \mathcal{L}(X, Y)$ .

Both, Gelfand and Kolmogorov numbers, are subadditive and multiplicative  $s$ -numbers, as well as approximation numbers. One may consult Pietsch [22](Sections 2.4, 2.5), for the proof in the Banach space case. And the generalization to  $p$ -Banach spaces follows obviously. Let  $Y$  be a  $p$ -Banach space,  $0 < p \leq 1$ . And let  $s_k$  denote any of the three quantities  $a_k$ ,  $d_k$  or  $c_k$ . Then we collect several common properties of them as follows,

(PS1) (monotonicity)  $\|T\| = s_1(T) \geq s_2(T) \geq \dots \geq 0$  for all  $T \in \mathcal{L}(X, Y)$ ,

(PS2) (subadditivity)  $s_{m+k-1}^p(S+T) \leq s_m^p(S) + s_k^p(T)$  for all  $m, k \in \mathbb{N}$ ,  $S, T \in \mathcal{L}(X, Y)$ ,

(PS3) (multiplicativity)  $s_{m+k-1}(ST) \leq s_m(S)s_k(T)$  for all  $T \in \mathcal{L}(X, Y)$ ,  $S \in \mathcal{L}(Y, Z)$

and  $m, k \in \mathbb{N}$ , cf. [21](p. 155), where  $Z$  denotes a quasi-Banach space,

(PS4) (rank property)  $\text{rank}(T) < k$  if and only if  $s_k(T) = 0$ , where  $T \in \mathcal{L}(X, Y)$ .

Moreover, there exist the following relationships:

$$c_k(T) \leq a_k(T), \quad d_k(T) \leq a_k(T), \quad k \in \mathbb{N}. \quad (2.3)$$

Now we recall the characterization of compactness of the embeddings under consideration, which was proved in [17].

**Proposition 2.2.** *Let  $U$  be a  $d$ -set,  $0 \leq d \leq n$ ,  $w_{i,j}(x) = (1 + 2^j \text{dist}(x, U))^{s'_i}$ ,  $i = 1, 2$ , and  $s' = s'_1 - s'_2 > 0$ . Then the embedding (1.3) is compact if and only if  $\delta > d/p^*$  and  $s' > n/p^*$ .*

For  $0 < p \leq \infty$ , we set

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty, \\ \infty & \text{if } 0 < p \leq 1. \end{cases}$$

We are now in position to state our main results.

**Theorem 2.3.** *Let  $U$  be a  $d$ -set,  $0 \leq d \leq n$ , and  $w_{i,j}(x) = (1 + 2^j \text{dist}(x, U))^{s'_i}$ ,  $i = 1, 2$ . Further, let  $t = \min(p'_1, p_2)$ ,  $s' = s'_1 - s'_2 > 0$  and  $\frac{1}{p} = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1}$ . We assume that  $0 < p_1 \leq p_2 \leq \infty$  or  $\tilde{p} < p_2 < p_1 \leq \infty$ .*

*Denote by  $a_k$  the  $k$ th approximation number of the embedding (1.3). Then  $a_k \sim k^{-\alpha}$ , where*

- (i)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right)$  if  $0 < p_1 \leq p_2 \leq 2$  or  $2 \leq p_1 \leq p_2 \leq \infty$ ,
- (ii)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2}$  if  $\tilde{p} < p_2 < p_1 \leq \infty$ ,
- (iii)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{2} - \frac{1}{t}$  if  $0 < p_1 < 2 < p_2 \leq \infty$  and  $\min\left(\frac{\delta}{d}, \frac{s'}{n}\right) > \frac{1}{t}$ ,
- (iv)  $\varkappa = \frac{s'}{n} \cdot \frac{t}{2}$  if  $0 < p_1 < 2 < p_2 \leq \infty$  and  $\delta > s'$ , with the following restrictions,
 
$$\begin{cases} \delta < \frac{d}{t}, \\ \delta - s' < \frac{2d-n}{t}, \end{cases} \quad \text{or} \quad \begin{cases} \delta + s' < \frac{n}{t}, \\ \delta - s' > \frac{2d-n}{t}, \end{cases}$$

Besides, suppose that in addition,  $0 < p_1 < 2 < p_2 \leq \infty$  and  $\delta < s'$ . Then

- (i)  $k^{-\frac{t}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \leq a_k \leq k^{-\frac{t}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}$  if  $\delta < \frac{d}{t}$ ,  $s' < \frac{n}{t}$  and  $\delta - s' < \frac{2d-n}{t}$ ;
- (ii)  $k^{-\frac{t}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \leq a_k \leq k^{-\frac{t}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}$  if  $\delta + s' < \frac{n}{t}$  and  $\delta - s' > \frac{2d-n}{t}$ .

**Remark 2.4.** Note that in the above assertion point (iv), as well as the latter statements (i) and (ii), vanishes if  $0 < p_1 \leq 1$  and  $p_2 = \infty$ .

**Theorem 2.5.** Let  $U$  be a  $d$ -set,  $0 \leq d \leq n$ , and  $w_{i,j}(x) = (1+2^j \text{dist}(x, U))^{s'_i}$ ,  $i = 1, 2$ . Further, let

$$s' = s'_1 - s'_2 > 0, \quad \theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2} \quad \text{and} \quad \frac{1}{\tilde{p}} = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1}.$$

We assume that  $0 < p_1 \leq p_2 \leq \infty$  or  $\tilde{p} < p_2 < p_1 \leq \infty$ .

Denote by  $d_k$  the  $k$ th Kolmogorov number of the embedding (1.3). Then  $d_k \sim k^{-\varkappa}$ , where

- (i)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right)$  if  $0 < p_1 \leq p_2 \leq 2$  or  $2 < p_1 = p_2 \leq \infty$ ,
- (ii)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2}$  if  $\tilde{p} < p_2 < p_1 \leq \infty$ ,
- (iii)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{2} - \frac{1}{p_2}$  if  $0 < p_1 < 2 < p_2 \leq \infty$  and  $\min\left(\frac{\delta}{d}, \frac{s'}{n}\right) > \frac{1}{p_2}$ ,
- (iv)  $\varkappa = \frac{s'}{n} \cdot \frac{p_2}{2}$  if  $0 < p_1 < 2 < p_2 < \infty$  and  $\delta > s'$ , with the following restrictions,
 
$$\begin{cases} \delta < \frac{d}{p_2}, \\ \delta - s' < \frac{2d-n}{p_2}, \end{cases} \quad \text{or} \quad \begin{cases} \delta + s' < \frac{n}{p_2}, \\ \delta - s' > \frac{2d-n}{p_2}. \end{cases}$$
- (v)  $\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2}$  if  $2 \leq p_1 < p_2 \leq \infty$  and  $\min\left(\frac{\delta}{d}, \frac{s'}{n}\right) > \frac{\theta}{p_2}$ ,
- (vi)  $\varkappa = \frac{s'}{n} \cdot \frac{p_2}{2}$  if  $2 \leq p_1 < p_2 < \infty$  and  $\delta > s'$ , with the following restrictions,
 
$$\begin{cases} \delta < \frac{d}{p_2} \theta, \\ \delta - s' < \frac{2d-n}{p_2} \theta, \end{cases} \quad \text{or} \quad \begin{cases} \delta + s' < \frac{n}{p_2} \theta, \\ \delta - s' > \frac{2d-n}{p_2} \theta. \end{cases}$$

Besides, we have the following statements.

- (i) Suppose that in addition,  $0 < p_1 < 2 < p_2 < \infty$  and  $\delta < s'$ . Then
  - (a)  $k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \leq d_k \leq k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}$  if  $\delta < \frac{d}{p_2}$ ,  $s' < \frac{n}{p_2}$  and  $\delta - s' < \frac{2d-n}{p_2}$ ;
  - (b)  $k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \leq d_k \leq k^{-\frac{p_2}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}$  if  $\delta + s' < \frac{n}{p_2}$  and  $\delta - s' > \frac{2d-n}{p_2}$ .

(ii) Suppose that in addition,  $2 \leq p_1 < p_2 < \infty$  and  $\delta < s'$ . Then

$$(a) \ k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq d_k \preceq k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})} \text{ if } \delta < \frac{d}{p_2}\theta, \ s' < \frac{n}{p_2}\theta \text{ and } \delta - s' < \frac{2d-n}{p_2}\theta;$$

$$(b) \ k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq d_k \preceq k^{-\frac{p_2}{n} \cdot \min(\delta, \frac{\delta+s'}{4})} \text{ if } \delta + s' < \frac{n}{p_2}\theta \text{ and } \delta - s' > \frac{2d-n}{p_2}\theta.$$

**Remark 2.6.** Points (iv) and (vi), as well as the latter statements (i) and (ii), vanish if  $p_2 = \infty$ .

**Theorem 2.7.** Let  $U$  be a  $d$ -set,  $0 \leq d \leq n$ , and  $w_{i,j}(x) = (1+2^j \text{dist}(x, U))^{s'_i}$ ,  $i = 1, 2$ . Further, let

$$s' = s'_1 - s'_2 > 0, \quad \theta_1 = \frac{1/p'_2 - 1/p'_1}{1/2 - 1/p'_1} \quad \text{and} \quad \frac{1}{\tilde{p}} = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1}.$$

We assume that  $0 < p_1 \leq p_2 \leq \infty$  or  $\tilde{p} < p_2 < p_1 \leq \infty$ .

Denote by  $c_k$  the  $k$ th Gelfand number of the embedding (1.3). Then  $c_k \sim k^{-\varkappa}$ , where

$$(i) \ \varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) \text{ if } 2 \leq p_1 \leq p_2 \leq \infty \text{ or } 0 < p_1 = p_2 < 2,$$

$$(ii) \ \varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2} \text{ if } \tilde{p} < p_2 < p_1 \leq \infty,$$

$$(iii) \ \varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{2} \text{ if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) > \frac{1}{p'_1},$$

$$(iv) \ \varkappa = \frac{s'}{n} \cdot \frac{p'_1}{2} \text{ if } 1 < p_1 < 2 < p_2 \leq \infty \text{ and } \delta > s', \text{ with the following restrictions,}$$

$$\begin{cases} \delta < \frac{d}{p'_1}, \\ \delta - s' < \frac{2d-n}{p'_1}, \end{cases} \quad \text{or} \quad \begin{cases} \delta + s' < \frac{n}{p'_1}, \\ \delta - s' > \frac{2d-n}{p'_1}. \end{cases}$$

$$(v) \ \varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2} \text{ if } 0 < p_1 < p_2 \leq 2 \text{ and } \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) > \frac{\theta_1}{p'_1},$$

$$(vi) \ \varkappa = \frac{s'}{n} \cdot \frac{p'_1}{2} \text{ if } 1 < p_1 < p_2 \leq 2 \text{ and } \delta > s', \text{ with the following restrictions,}$$

$$\begin{cases} \delta < \frac{d}{p'_1}\theta_1, \\ \delta - s' < \frac{2d-n}{p'_1}\theta_1, \end{cases} \quad \text{or} \quad \begin{cases} \delta + s' < \frac{n}{p'_1}\theta_1, \\ \delta - s' > \frac{2d-n}{p'_1}\theta_1. \end{cases}$$

Besides, we have the following statements.

(i) Suppose that in addition,  $1 < p_1 < 2 < p_2 \leq \infty$  and  $\delta < s'$ . Then

$$(a) \ k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \preceq k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})} \text{ if } \delta < \frac{d}{p'_1}, \ s' < \frac{n}{p'_1} \text{ and } \delta - s' < \frac{2d-n}{p'_1},$$

$$(b) \ k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \preceq k^{-\frac{p'_1}{n} \cdot \min(\delta, \frac{\delta+s'}{4})} \text{ if } \delta + s' < \frac{n}{p'_1} \text{ and } \delta - s' > \frac{2d-n}{p'_1}.$$

(ii) Suppose that in addition,  $1 < p_1 < p_2 \leq 2$  and  $\delta < s'$ . Then

$$(a) \ k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \preceq k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})} \text{ if } \delta < \frac{d}{p'_1}\theta_1, \ s' < \frac{n}{p'_1}\theta_1 \text{ and } \delta - s' < \frac{2d-n}{p'_1}\theta_1,$$

$$(b) \ k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \preceq k^{-\frac{p'_1}{n} \cdot \min(\delta, \frac{\delta+s'}{4})} \text{ if } \delta + s' < \frac{n}{p'_1}\theta_1 \text{ and } \delta - s' > \frac{2d-n}{p'_1}\theta_1.$$

**Remark 2.8.** Points (iv) and (vi), together with the latter statements (i) and (ii), vanish if  $0 < p_1 \leq 1$ .

**Remark 2.9.** We shift the proofs of the above three theorems to Section 4.

Now, we wish to compare the approximation, Gelfand and Kolmogorov numbers of the embedding (1.3). The comparison of these above results shows that

- (i)  $a_n \sim c_n$  if either
  - (a)  $2 \leq p_1 < p_2 \leq \infty$  or,
  - (b)  $\tilde{p} < p_2 \leq p_1 \leq \infty$  or,
  - (c)  $1 \leq p_1 < p'_1 \leq p_2 \leq \infty$  and  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{p_1}$ ;
- (ii)  $a_n \sim d_n$  if either
  - (a)  $0 < p_1 < p_2 \leq 2$  or,
  - (b)  $\tilde{p} < p_2 \leq p_1 \leq \infty$  or,
  - (c)  $0 < p_1 < 2 < p_2 \leq p'_1 \leq \infty$  and  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{p_2}$ ;
- (iii)  $c_n \sim d_n$  if either
  - (a)  $\tilde{p} < p_2 \leq p_1 \leq \infty$  or,
  - (b)  $1 \leq p_1 < p'_1 = p_2 \leq \infty$  and  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{p_2}$ .

Note that we don't discuss above the case when  $0 < p_1 < 2 < p_2 \leq \infty$  and  $\min(\frac{\delta}{d}, \frac{s'}{n}) < \frac{1}{\min(p'_1, p_2)}$ .

### 3 Widths in sequence spaces

This section is the heart of the paper. Our main aim will be to determine the asymptotic behaviour of related widths of compact embeddings between weighted sequence spaces  $\ell_q(2^{js}\ell_p(w))$ , where the sequences are indexed by  $\mathbb{N}_0 \times \mathbb{Z}^n$ .

#### 3.1 Preliminaries

Here we are going to use the discrete wavelet transform in order to obtain equivalent quasi-norms in the spaces  $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$  which, in a quite natural way, will establish isomorphism between  $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$  and appropriate sequence spaces, cf. [10]. By now this is a standard method to reduce complicated problems in function spaces to simpler problems in sequence spaces. The key point in this discretization technique is that the asymptotic order of the estimates is preserved.

Wavelet bases in function spaces are a well-developed concept, see [28] for a survey. We adopt the notation from [27] (Section 4.2.1) with  $l = 0$ . Let  $\psi_M, \psi_F \in C^k(\mathbb{R})$  be real compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} x^\beta \psi_M(x) dx = 0 \quad \text{for } |\beta| < k.$$

Let  $G^0 = \{F, M\}^n$  and let  $G^j = \{F, M\}^{n^*}$  where  $n^*$  indicates that at least one  $G_i$  of  $G = (G_1, \dots, G_n) \in \{F, M\}^{n^*}$  must be an  $M$ . It is clear that the cardinal number of  $\{F, M\}^{n^*}$  is  $2^n - 1$ . Let for  $x \in \mathbb{R}^n$

$$\psi_{Gm}^j(x) = 2^{j\frac{n}{2}} \prod_{i=1}^n \psi_{G_i}(2^j x_i - m_i) \quad \text{where } j \in \mathbb{N}_0, m \in \mathbb{Z}^n \text{ and } G = (G_1, \dots, G_n) \in G^j.$$

Then  $\{\psi_{Gm}^j : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\}$  is an orthonormal basis in  $L_2(\mathbb{R}^n)$ , see [10, 27].

In our situation we shall consider the following sequence spaces, see [10] (Section 5.3.4).

**Definition 3.1.** Let  $w = (w_j)_{j \in \mathbb{N}_0}$  be as in (1.2). We put

$$\tilde{b}_{p,q}^{s,s'}(w) = \left\{ (\lambda_{Gm}^j)_{j,G,m} : \|\lambda\|_{\tilde{b}_{p,q}^{s,s'}} < \infty \right\},$$

where

$$\|\lambda\|_{\tilde{b}_{p,q}^{s,s'}} = \left( \sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{Gm}^j|^p w_j^p(2^{-j}m) \right)^{q/p} \right)^{1/q}.$$

The following proposition can also be found there, see [10] (Corollary 5.33); cf. also [12, 17].

**Proposition 3.2.** Let  $U \in \mathbb{R}^n$  bounded and  $w = (w_j)_{j \in \mathbb{N}_0}$  as in (1.2). Further, let  $f \in B_{p,q}^{s,s'}(\mathbb{R}^n, U)$ ,  $k$  large enough and

$$\lambda_{Gm}^j = \lambda_{Gm}^j(f) = 2^{j\frac{n}{2}} \langle f, \psi_{Gm}^j \rangle = 2^{j\frac{n}{2}} \int f(x) \psi_{Gm}^j dx.$$

Then

$$I : f \mapsto 2^{j\frac{n}{2}} \langle f, \psi_{Gm}^j \rangle$$

is an isomorphic map from  $B_{p,q}^{s,s'}(\mathbb{R}^n, U)$  onto  $\tilde{b}_{p,q}^{s,s'}(w)$ .

Inspired by Proposition 3.2, we shall work with the following weighted sequence spaces. Let  $(w_j)_{j \in \mathbb{N}_0}$  be a given weight sequence as in (1.2). We put

$$\begin{aligned} \ell_q(2^{js} \ell_p(w)) &:= \left\{ \lambda = (\lambda_{j,m})_{j,m} : \lambda_{j,m} \in \mathbb{C}, \right. \\ &\quad \left. \|\lambda\|_{\ell_q(2^{js} \ell_p(w))} = \left( \sum_{j=0}^{\infty} 2^{jsq} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m} w_j(2^{-j}m)|^p \right)^{q/p} \right)^{1/q} < \infty \right\}, \end{aligned} \quad (3.1)$$

(usual modification if  $p = \infty$  and/or  $q = \infty$ ). If  $s = 0$  we will write  $\ell_q(\ell_p(w))$ . In contrast to the quasi-norm defined in Definition 3.1, the finite summation on  $G \in G^j$  is irrelevant and can be omitted. Similar considerations may be found in [7, 16, 17, 24]. Furthermore, let

$$\begin{aligned} A_1 &:= \ell_{q_1}(2^{j(s_1-\frac{n}{p_1})} \ell_{p_1}(v_1)), & A_2 &:= \ell_{q_2}(2^{j(s_2-\frac{n}{p_2})} \ell_{p_2}(v_2)), \\ B_1 &:= \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), & B_2 &:= \ell_{q_2}(\ell_{p_2}), \end{aligned}$$

where

$$v_1 = (w_{1,j})_{j \in \mathbb{N}_0}, \quad v_2 = (w_{2,j})_{j \in \mathbb{N}_0} \quad \text{and} \quad w = (w_j)_{j \in \mathbb{N}_0} \quad \text{with} \quad w_j(x) = \frac{w_{1,j}(x)}{w_{2,j}(x)}, \quad \text{for } x \in \mathbb{Z}^n.$$

As is already discussed in [17], we observe by the properties of  $s$ -numbers that

$$s_k(\text{id}, A_1, A_2) = s_k(\text{id}, B_1, B_2), \quad k = 1, 2, \dots, \quad (3.2)$$



where  $s_k$  denotes any of the three quantities  $a_k$ ,  $d_k$  or  $c_k$ . Hence, we may concentrate on  $s_k(\text{id}, B_1, B_2)$ .

We shall consider compact subsets of  $\mathbb{R}^n$  for  $U$ , in order to guarantee the compactness of the embedding  $B_1 \hookrightarrow B_2$ . We are now able to recall the definition of  $d$ -sets, which are fractal sets in between single point sets  $\{x_0\}$  and compact sets with non-empty interior, cf. [26].

Let  $U$  be a compact set and  $\mu$  a Radon measure with  $\text{supp}\mu = U$ . The set  $U$  is called a  $d$ -set,  $0 \leq d \leq n$ , if for each ball of radius  $r$  and centered in  $\gamma \in U$  holds

$$\mu(B(\gamma, r)) \sim r^d, \quad \text{with } 0 < r < 1.$$

Let us mention the estimation of a number of dyadic cubes of a fixed side length that are in a predetermined distance to the set  $U$ .

For  $i, j \in \mathbb{N}_0$ , we denote by  $N_{j,i}$  the number of cubes  $Q_{j,\ell}$  of side length  $2^{-j}$ , centered in  $2^{-j}\ell$  with

$$\sqrt{n}2^{-j+i} < \text{dist}(Q_{j,\ell}, U) \leq 4\sqrt{n}2^{-j+i}, \quad \ell \in \mathbb{Z}^n. \quad (3.3)$$

The following lemma may be found in [17].

**Lemma 3.3.** *Let  $U$  be a  $d$ -set, then*

$$N_{j,i} \sim \begin{cases} 2^{in}2^{(j-i)d} & 0 \leq i < j, \\ 2^{in} & j \leq i. \end{cases}$$

Following Pietsch [22], we associate to the sequence of the  $s$ -numbers the following operator ideals, and for  $0 < r < \infty$ , we put

$$\mathcal{L}_{r,\infty}^{(s)} := \left\{ T \in \mathcal{L}(X, Y) : \sup_{n \in \mathbb{N}} n^{1/r} s_n(T) < \infty \right\}. \quad (3.4)$$

Equipped with the quasi-norm

$$L_{r,\infty}^{(s)}(T) := \sup_{n \in \mathbb{N}} n^{1/r} s_n(T), \quad (3.5)$$

the set  $\mathcal{L}_{r,\infty}^{(s)}$  becomes a quasi-Banach space. For such quasi-Banach spaces there always exists a real number  $0 < \rho \leq 1$  such that

$$L_{r,\infty}^{(s)} \left( \sum_j T_j \right)^\rho \leq \sum_j L_{r,\infty}^{(s)}(T_j)^\rho \quad (3.6)$$

holds for any sequence of operators  $T_j \in \mathcal{L}_{r,\infty}^{(s)}$ . Then we shall use the quasi-norms  $L_{r,\infty}^{(a)}$ ,  $L_{r,\infty}^{(c)}$  and  $L_{r,\infty}^{(d)}$  for the approximation, Gelfand and Kolmogorov numbers, respectively.

**Remark 3.4.** *We would like to add some comments on operator ideals. Historically, the technique of estimating single  $s$ -numbers (or entropy numbers) via estimates of ideal (quasi-)norms derives from ideas of Carl [2]. In the 1980s this technique was frequently used in operator theory, in eigenvalue problems for Banach space operators, etc. In the function spaces community however, the operator ideal technique remained unknown for many years. As far as we know, it was applied for the first time in [14, 15], both of which appeared in 2003.*

For brevity's sake, we wish to make an additional agreement throughout the following three subsections. Let  $U$  be a  $d$ -set,  $0 \leq d \leq n$ , and  $w_{j,\ell} = w_j(2^{-j}\ell) = (1 + 2^j \text{dist}(2^{-j}\ell, U))^{s'}$  a sequence of weights,  $j \in \mathbb{N}_0$ ,  $\ell \in \mathbb{Z}^n$ ,  $s' = s'_1 - s'_2 > 0$ , if no further restrictions are stated.

### 3.2 Approximation numbers of sequence spaces

To begin with, we shall recall some lemmata. Lemma 3.5 follows trivially from results of Gluskin [6] and Edmunds and Triebel [3]. Lemma 3.6 is due to Vybíral [29].

**Lemma 3.5.** *Let  $N \in \mathbb{N}$  and  $k \leq \frac{N}{4}$ .*

(i) *If  $0 < p_1 \leq p_2 \leq 2$  or  $2 \leq p_1 \leq p_2 \leq \infty$  then*

$$a_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim 1.$$

(ii) *If  $1 \leq p_1 < 2 < p_2 \leq \infty$  and  $(p_1, p_2) \neq (1, \infty)$  then*

$$a_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim \min(1, N^{1/t} k^{-1/2}).$$

$$\text{where } \frac{1}{t} = \frac{1}{\min(p'_1, p_2)}.$$

(iii) *If  $0 < p_1 < 1$  and  $2 < p_2 < \infty$  then*

$$a_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim \min(1, N^{1/p_2} k^{-1/2}).$$

**Lemma 3.6.** *Let  $0 < p \leq 1$  and  $N \in \mathbb{N}$ .*

(i) *Let  $0 < \lambda < 1$ . Then there exists a constant  $c_\lambda > 0$  depending only on  $\lambda$  such that*

$$a_k(\text{id}, \ell_p^N, \ell_\infty^N) \leq \begin{cases} 1 & \text{if } k \leq N^\lambda, \\ c_\lambda k^{-1/2} & \text{if } N^\lambda < k \leq N, \\ 0 & \text{if } k > N. \end{cases} \quad (3.7)$$

(ii) *There exists a constant  $C > 0$  independent of  $k$  such that for any  $k \in \mathbb{N}$*

$$a_k(\text{id}, \ell_p^{2k}, \ell_\infty^{2k}) \geq C n^{-1/2}. \quad (3.8)$$

Lemma 3.7 in the case  $1 \leq p_2 < p_1 \leq \infty$  may be found in Pietsch [21], Section 11.11.5, also in Pinkus [23](p. 203). The proof may be directly generalized to the quasi-Banach setting  $0 < p_2 < p_1 \leq \infty$ .

**Lemma 3.7.** *Let  $0 < p_2 < p_1 \leq \infty$  and  $k \leq N$ . Then*

$$a_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) = (N - k + 1)^{1/p_2 - 1/p_1}.$$

The following lemma in the case  $1 \leq p_1 < 2 < p_2 \leq \infty$  may be found in [24]. The proof may be trivially extended to the quasi-Banach spaces with  $0 < p_1 < 2 < p_2 \leq \infty$ , by virtue of Lemma 3.5 (iii).

**Lemma 3.8.** Suppose  $0 < p_1 < 2 < p_2 \leq \infty$ , and assume that  $p_2 \neq \infty$  when  $0 < p_1 \leq 1$ . Then there is a positive constant  $C$  independent of  $N$  and  $k$  such that

$$a_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \leq C \begin{cases} 1 & \text{if } k \leq N^{2/t}, \\ N^{1/t} k^{-1/2} & \text{if } N^{2/t} < k \leq N, \\ 0 & \text{if } k > N, \end{cases} \quad (3.9)$$

where  $\frac{1}{t} = \frac{1}{\min(p'_1, p_2)}$ .

**Proposition 3.9.** Suppose  $0 < p_1 < 2 < p_2 \leq \infty$  and  $t = \min(p'_1, p_2)$ .

(i) If  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{t}$ , then

$$a_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\min(\frac{\delta}{d}, \frac{s'}{n}) + \frac{1}{t} - \frac{1}{2}}. \quad (3.10)$$

(ii) If  $\delta > s'$  and either  $\begin{cases} \delta < \frac{d}{t}, \\ \delta - s' < \frac{2d-n}{t}, \end{cases}$  or  $\begin{cases} \delta + s' < \frac{n}{t}, \\ \delta - s' > \frac{2d-n}{t}, \end{cases}$  then

$$a_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\frac{s'}{n} \cdot \frac{t}{2}}. \quad (3.11)$$

(iii) If  $\delta < \frac{d}{t}$ ,  $\delta < s' < \frac{n}{t}$  and  $\delta - s' < \frac{2d-n}{t}$ , then

$$k^{-\frac{t}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq a_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{t}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}. \quad (3.12)$$

(iv) If  $\delta < s'$ ,  $\delta + s' < \frac{n}{t}$  and  $\delta - s' > \frac{2d-n}{t}$ , then

$$k^{-\frac{t}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq a_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{t}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}. \quad (3.13)$$

*Proof. Step 1.* Preparations. We denote

$$\Lambda := \{\lambda = (\lambda_{j,\ell}) : \lambda_{j,\ell} \in \mathbb{C}, \quad j \in \mathbb{N}_0, \ell \in \mathbb{Z}^n\},$$

and set

$$B_1 = \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \quad \text{and} \quad B_2 = \ell_{q_2}(\ell_{p_2}).$$

Let  $I_{j,i} \subset \mathbb{N}_0 \times \mathbb{Z}^n$  be such that

$$I_{j,0} := \{(j, \ell) : \text{dist}(2^{-j}\ell, U) \leq \sqrt{n}2^{-j}\}, \quad j \in \mathbb{N}_0, \quad (3.14)$$

$$I_{j,i} := \{(j, \ell) : \sqrt{n}2^{-j+i-1} < \text{dist}(2^{-j}\ell, U) \leq \sqrt{n}2^{-j+i}\}, \quad i \in \mathbb{N}, \quad j \in \mathbb{N}_0. \quad (3.15)$$

Besides, let  $P_{j,i} : \Lambda \mapsto \Lambda$  be the canonical projection with respect to  $I_{j,i}$ , i.e., for  $\lambda \in \Lambda$ , we put

$$(P_{j,i}\lambda)_{u,v} := \begin{cases} \lambda_{u,v} & (u, v) \in I_{j,i}, \\ 0 & \text{otherwise,} \end{cases} \quad u \in \mathbb{N}_0, \quad v \in \mathbb{Z}^n, \quad i \geq 0.$$

Then

$$\text{id}_\Lambda = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} P_{j,i}. \quad (3.16)$$

$$w_j(2^{-j}\ell) \sim (1 + 2^j 2^{-j+i})^{s'} \sim 2^{is'} \quad \text{if } (j, \ell) \in I_{j,i}, \quad i \geq 0. \quad (3.17)$$

Due to Lemma 3.3 and the structure of  $U$ , cf. [17], we have

$$M_{j,i} := |I_{j,i}| \leq N_{j,i+2} + N_{j,i+3} \sim \begin{cases} 2^{in} 2^{(j-i)d}, & 0 \leq i < j, \\ 2^{in}, & 0 \leq j \leq i, \end{cases} \quad (3.18)$$

and

$$M_{j,i+1} + M_{j,i+2} \geq N_{j,i} \sim \begin{cases} 2^{in} 2^{(j-i)d}, & 0 \leq i < j, \\ 2^{in}, & 0 \leq j \leq i. \end{cases} \quad (3.19)$$

Thanks to simple monotonicity arguments and explicit properties of the approximation numbers, there is a positive constant  $C$  independent of  $k$ ,  $j$  and  $i$  such that

$$a_k(P_{j,i}, B_1, B_2) \leq C 2^{-j\delta} 2^{-is'} a_k(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}). \quad (3.20)$$

**Step 2.** The operator ideal plays an important role. To shorten notations we shall put  $\frac{1}{s} = \frac{1}{r} + \frac{1}{2}$  for any  $s > 0$ . By (3.5) and (3.20), we have

$$L_{s,\infty}^{(a)}(P_{j,i}) \leq C 2^{-j\delta} 2^{-is'} L_{s,\infty}^{(a)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \quad (3.21)$$

The known asymptotic behavior of the approximation numbers  $a_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N)$ , cf. (3.9), and (3.18) yield that, with the assumption  $p_2 \neq \infty$  if  $0 < p_1 \leq 1$ ,

$$L_{2,\infty}^{(a)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \leq C \begin{cases} 2^{(in+d(j-i))/t}, & 0 \leq i < j, \\ 2^{\frac{in}{t}}, & 0 \leq j \leq i, \end{cases} \quad (3.22)$$

$$L_{s,\infty}^{(a)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \leq C \begin{cases} 2^{(in+d(j-i))(\frac{1}{t} + \frac{1}{r})}, & 0 \leq i < j, \quad \frac{1}{s} > \frac{1}{2}, \\ 2^{in(\frac{1}{t} + \frac{1}{r})}, & 0 \leq j \leq i, \quad \frac{1}{s} > \frac{1}{2}, \end{cases} \quad (3.23)$$

and in consequence

$$L_{2,\infty}^{(a)}(P_{j,i}) \leq C 2^{-j\delta} 2^{-is'} \begin{cases} 2^{(in+d(j-i))/t}, & 0 \leq i < j, \\ 2^{\frac{in}{t}}, & 0 \leq j \leq i, \end{cases} \quad (3.24)$$

$$L_{s,\infty}^{(a)}(P_{j,i}) \leq C 2^{-j\delta} 2^{-is'} \begin{cases} 2^{(in+d(j-i))(\frac{1}{t} + \frac{1}{r})}, & 0 \leq i < j, \quad \frac{1}{s} > \frac{1}{2}, \\ 2^{in(\frac{1}{t} + \frac{1}{r})}, & 0 \leq j \leq i, \quad \frac{1}{s} > \frac{1}{2}. \end{cases} \quad (3.25)$$

If  $0 < p_1 \leq 1$  and  $p_2 = \infty$ , we have  $t = \infty$  and select  $0 < \lambda < 1$  such that  $\frac{\lambda}{2(1-\lambda)} < \min(\frac{\delta}{d}, \frac{s'}{n})$ . The inequality  $\lambda \cdot \frac{1}{s} \leq \frac{1}{s} - \frac{1}{2}$  holds if and only if  $\frac{1}{s} \geq \frac{1}{2(1-\lambda)}$ , where  $0 < \lambda < 1$ . So there exists a

constant  $s > 0$  such that  $\lambda \cdot \frac{1}{s} < \frac{1}{s} - \frac{1}{2} < \min\left(\frac{\delta}{d}, \frac{s'}{n}\right)$ . Then, in a similar way as above, we find by (3.7) that

$$L_{s,\infty}^{(a)}(P_{j,i}) \leq C 2^{-j\delta} 2^{-is'} \begin{cases} 2^{(in+d(j-i))\frac{1}{r}}, & 0 \leq i < j, \frac{1}{s} > \frac{1}{2(1-\lambda)}, \\ 2^{\frac{in}{r}}, & 0 \leq j \leq i, \frac{1}{s} > \frac{1}{2(1-\lambda)}. \end{cases} \quad (3.26)$$

**Step 3.** The estimate of  $a_k(\text{id}, B_1, B_2)$  from above in case (i),  $\min\left(\frac{\delta}{d}, \frac{s'}{n}\right) > \frac{1}{t}$ . Let  $M \in \mathbb{N}_0$  be given. For the identity operator  $\text{id}_\Lambda$ , we use the same division, as in the proof of Theorem 6 in [17],

$$\begin{aligned} P^1 &:= \sum_{j=0}^M \sum_{i=0}^{j-1} P_{j,i}, & P^2 &:= \sum_{j=M+1}^{\infty} \sum_{i=0}^{j-1} P_{j,i}, \\ Q^1 &:= \sum_{j=0}^M \sum_{i=j}^M P_{j,i}, & Q^2 &:= \sum_{j=0}^M \sum_{i=M+1}^{\infty} P_{j,i}, & Q^3 &:= \sum_{j=M+1}^{\infty} \sum_{i=j}^{\infty} P_{j,i}. \end{aligned} \quad (3.27)$$

Next, the proof in this case follows literally the presentation given in [17] with the obvious changes, now using  $1/r + 1/t$  instead of  $1/r - 1/p$ , where the latter notations derive from [17]. So we don't expand here.

**Step 4.** Now let  $\delta \neq s'$ ,  $s' < n/t$  and  $\delta - s' \neq \frac{2d-n}{t}$ . We turn to using the following division

$$\begin{aligned} \text{id} &= \sum_{m=0}^{M_1} \sum_{\substack{i+j=m \\ i < j}} P_{j,i} + \sum_{m=M_1+1}^{M_2} \sum_{\substack{i+j=m \\ i < j}} P_{j,i} + \sum_{m=M_2+1}^{\infty} \sum_{\substack{i+j=m \\ i < j}} P_{j,i} \\ &+ \sum_{m=0}^{M_3} \sum_{\substack{i+j=m \\ i \geq j}} P_{j,i} + \sum_{m=M_3+1}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} P_{j,i} + \sum_{m=M_4+1}^{\infty} \sum_{\substack{i+j=m \\ i \geq j}} P_{j,i}, \end{aligned} \quad (3.28)$$

where  $M_1, M_2, M_3, M_4 \in \mathbb{N}$ ,  $M_1 < M_2$  and  $M_3 < M_4$ , which will be determined later on for given  $k \in \mathbb{N}$ . In terms of the subadditivity of  $s$ -numbers, we observe

$$a_{k'}(\text{id}, B_1, B_2) \leq \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6, \quad (3.29)$$

where

$$\begin{aligned} \Delta_1 &= \sum_{m=0}^{M_1} \sum_{\substack{i+j=m \\ i < j}} a_{k_{j,i}}(P_{j,i}), & \Delta_2 &= \sum_{m=M_1+1}^{M_2} \sum_{\substack{i+j=m \\ i < j}} a_{k_{j,i}}(P_{j,i}), & \Delta_3 &= \sum_{m=M_2+1}^{\infty} \sum_{\substack{i+j=m \\ i < j}} \|P_{j,i}\|, \\ \Delta_4 &= \sum_{m=0}^{M_3} \sum_{\substack{i+j=m \\ i \geq j}} a_{k_{j,i}}(P_{j,i}), & \Delta_5 &= \sum_{m=M_3+1}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} a_{k_{j,i}}(P_{j,i}), & \Delta_6 &= \sum_{m=M_4+1}^{\infty} \sum_{\substack{i+j=m \\ i \geq j}} \|P_{j,i}\|, \end{aligned}$$

and

$$k' - 1 = \sum_{m=0}^{M_2} \sum_{\substack{i+j=m \\ i < j}} (k_{j,i} - 1) + \sum_{m=0}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} (k_{j,i} - 1).$$

**Substep 4.1.** First we deal with those parts concerning  $i < j$  (i.e.,  $\triangle_1, \triangle_2$  and  $\triangle_3$ ). We take

$$M_1 = \begin{cases} \left\lfloor \frac{\log_2 k}{n/2} \right\rfloor, & \text{if } 2d < n, \\ \left\lfloor \frac{\log_2 k}{d} \right\rfloor, & \text{if } 2d > n, \\ \left\lfloor \frac{\log_2 k}{d} - \frac{\log_2 \log_2 k}{d} \right\rfloor, & \text{if } 2d = n, \end{cases} \quad \text{and} \quad M_2 = \begin{cases} \left\lfloor \frac{t}{2} \cdot \frac{\log_2 k}{d} \right\rfloor, & \text{if } \delta - s' < \frac{2d-n}{t}, \\ \left\lfloor \frac{t}{2} \cdot \frac{\log_2 k}{n/2} \right\rfloor, & \text{if } \delta - s' > \frac{2d-n}{t}, \end{cases}$$

where  $[a]$  denotes the largest integer smaller than  $a \in \mathbb{R}$  and  $\log_2 k$  is a dyadic logarithm of  $k$ . Then

$$\begin{aligned} \triangle_3 &= \sum_{m=M_2+1}^{\infty} \sum_{\substack{i+j=m \\ i < j}} \|P_{j,i}\| \leq c_1 \sum_{m=M_2+1}^{\infty} \sum_{\substack{i+j=m \\ i < j}} 2^{-j\delta} 2^{-is'} \\ &\leq \begin{cases} c_2 \sum_{m=M_2+1}^{\infty} 2^{-m\delta} \leq c_3 2^{-M_2\delta}, & \text{if } \delta < s', \\ c_2 \sum_{m=M_2+1}^{\infty} 2^{-\frac{m}{2}(\delta+s')} \leq c_3 2^{-M_2 \frac{\delta+s'}{2}}, & \text{if } \delta > s'. \end{cases} \end{aligned}$$

Next, we choose proper  $k_{j,i}$  for estimating  $\triangle_1$  and  $\triangle_2$ . If  $i < j$  and  $i + j \leq M_1$ , we take  $k_{j,i} = M_{j,i} + 1$  such that  $a_{k_{j,i}}(P_{j,i}) = 0$  and  $\triangle_1 = 0$ . And we obtain

$$\sum_{m=0}^{M_1} \sum_{\substack{i+j=m \\ i < j}} k_{j,i} \leq c_1 \sum_{m=0}^{M_1} \sum_{\substack{i+j=m \\ i < j}} 2^{dj} 2^{(n-d)i} \leq \begin{cases} c_2 \sum_{m=0}^{M_1} 2^{m \frac{n}{2}} \leq c_3 2^{M_1 \frac{n}{2}} \leq c_3 k, & \text{if } 2d < n, \\ c_2 \sum_{m=0}^{M_1} 2^{md} \leq c_3 2^{M_1 d} \leq c_3 k, & \text{if } 2d > n, \\ c_2 \sum_{m=0}^{M_1} \frac{m}{2} 2^{md} \leq c_3 M_1 \cdot 2^{M_1 d} \leq c_4 k, & \text{if } 2d = n. \end{cases}$$

Now we give the crucial choice of  $k_{j,i}$  for  $\triangle_2$ . We put

$$k_{j,i} = [k^{1-\varepsilon} \cdot 2^{iz_1} \cdot 2^{jz_2}],$$

where  $\varepsilon, z_1, z_2$  are positive real numbers such that

$$\delta + \frac{z_2}{2} < \frac{d}{t}, \quad 0 < \frac{z_2 - z_1}{2} < s' - \delta + \frac{2d-n}{t} \quad \text{and} \quad \frac{z_2 t}{2d} = \varepsilon \quad \text{if } \delta < \frac{d}{t} \text{ and } \delta - s' < \frac{2d-n}{t},$$

or

$$\delta + s' + \frac{z_1 + z_2}{2} < \frac{n}{t}, \quad 0 < \frac{z_1 - z_2}{2} < \delta - s' + \frac{n-2d}{t}, \quad \frac{(z_1 + z_2)t}{2n} = \varepsilon \quad \text{if } \delta + s' < \frac{n}{t} \text{ and } \delta - s' > \frac{2d-n}{t}.$$

Note that the relation,  $0 < \varepsilon < 1$ , holds obviously. We observe that

$$\begin{aligned} \sum_{m=M_1+1}^{M_2} \sum_{\substack{i+j=m \\ i < j}} k_{j,i} &\leq c_1 k^{1-\varepsilon} \sum_{m=M_1+1}^{M_2} \sum_{\substack{i+j=m \\ i < j}} 2^{iz_1} \cdot 2^{jz_2} \\ &\leq \begin{cases} c_2 k^{1-\varepsilon} \sum_{m=M_1+1}^{M_2} 2^{mz_2} \leq c_3 k^{1-\varepsilon} 2^{M_2 z_2} = c_3 k, & \text{if } \delta < \frac{d}{t} \text{ and } \delta - s' < \frac{2d-n}{t}, \\ c_2 k^{1-\varepsilon} \sum_{m=M_1+1}^{M_2} 2^{m \frac{z_1+z_2}{2}} \leq c_3 k^{1-\varepsilon} 2^{M_2 \frac{z_1+z_2}{2}} = c_3 k, & \text{if } \delta + s' < \frac{n}{t} \text{ and } \delta - s' > \frac{2d-n}{t}, \end{cases} \end{aligned}$$

and, in terms of (3.24),

$$\begin{aligned} \Delta_2 &\leq c_1 \sum_{m=M_1+1}^{M_2} \sum_{\substack{i+j=m \\ i < j}} 2^{-j\delta - is'} 2^{(in+d(j-i))/t} [k^{1-\varepsilon} \cdot 2^{iz_1} \cdot 2^{jz_2}]^{-\frac{1}{2}} \\ &\leq c_2 k^{-\frac{1}{2}(1-\varepsilon)} \sum_{m=M_1+1}^{M_2} \sum_{\substack{i+j=m \\ i < j}} 2^{-j(\delta - \frac{d}{t} + \frac{z_2}{2})} 2^{-i(s' - \frac{n-d}{t} + \frac{z_1}{2})} \\ &\leq \begin{cases} c_3 k^{-\frac{1}{2}(1-\varepsilon)} \sum_{m=M_1+1}^{M_2} 2^{-m(\delta - \frac{d}{t} + \frac{z_2}{2})} \leq c_4 2^{-M_2 \delta} = c_4 k^{-\frac{t\delta}{2d}}, & \text{if } \delta < \frac{d}{t} \text{ and } \delta - s' < \frac{2d-n}{t}, \\ c_3 k^{-\frac{1}{2}(1-\varepsilon)} \sum_{m=M_1+1}^{M_2} 2^{-\frac{m}{2}(\delta + s' + \frac{z_1+z_2}{2} - \frac{n}{t})} \leq c_4 k^{-\frac{t(\delta+s')}{2n}}, & \text{if } \delta + s' < \frac{n}{t} \text{ and } \delta - s' > \frac{2d-n}{t}. \end{cases} \end{aligned}$$

**Substep 4.2.** We consider those parts concerning  $i \geq j$  (i.e.,  $\Delta_4, \Delta_5$  and  $\Delta_6$ ). We take

$$M_3 = \left\lceil \frac{\log_2 k}{n} \right\rceil \quad \text{and} \quad M_4 = \left\lceil \frac{t}{2} \cdot \frac{\log_2 k}{n} \right\rceil.$$

It should be noted that the inequalities,  $M_1 \neq M_3$  and  $M_2 \neq M_4$ , hold in general. Then

$$\begin{aligned} \Delta_6 &= \sum_{m=M_2+1}^{\infty} \sum_{\substack{i+j=m \\ i \geq j}} \|P_{j,i}\| \leq c_1 \sum_{m=M_2+1}^{\infty} \sum_{\substack{i+j=m \\ i \geq j}} 2^{-j\delta} 2^{-is'} \\ &\leq \begin{cases} c_2 \sum_{m=M_2+1}^{\infty} 2^{-\frac{m}{2}(\delta+s')} \leq c_3 2^{-M_4 \frac{\delta+s'}{2}} \leq c_3 k^{-\frac{t(\delta+s')}{4n}}, & \text{if } \delta < s', \\ c_2 \sum_{m=M_2+1}^{\infty} 2^{-ms'} \leq c_3 2^{-M_4 s'} \leq c_3 k^{-\frac{ts'}{2n}}, & \text{if } \delta > s'. \end{cases} \end{aligned}$$

If  $i \geq j$  and  $i+j \leq M_3$ , we take  $k_{j,i} = M_{j,i} + 1$  such that  $a_{k_{j,i}}(P_{j,i}) = 0$  and  $\Delta_4 = 0$ . Moreover,

$$\sum_{m=0}^{M_3} \sum_{\substack{i+j=m \\ i \geq j}} k_{j,i} \leq c_1 \sum_{m=0}^{M_3} \sum_{\substack{i+j=m \\ i \geq j}} 2^{ni} \leq c_2 \sum_{m=0}^{M_3} 2^{mn} \leq c_3 2^{M_3 n} \leq c_4 k.$$

For  $\Delta_5$ , we put similarly

$$k_{j,i} = [k^{1-\varepsilon} \cdot 2^{iz_3} \cdot 2^{jz_4}],$$

where  $\varepsilon, z_3, z_4$  are positive real numbers such that  $s' + \frac{z_3}{2} < \frac{n}{t}$ ,  $0 < \frac{z_3 - z_4}{2} < \delta - s' + \frac{n}{t}$  and  $\frac{z_3 t}{2n} = \varepsilon$ . Recall that  $s' < n/t$ . And observe that

$$\sum_{m=M_3+1}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} k_{j,i} \leq c_1 k^{1-\varepsilon} \sum_{m=M_3+1}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} 2^{iz_3} \cdot 2^{jz_4} \leq c_2 k^{1-\varepsilon} \sum_{m=M_3+1}^{M_4} 2^{mz_3} \leq c_3 k^{1-\varepsilon} 2^{M_4 z_3} = c_3 k,$$

and, in terms of (3.24),

$$\begin{aligned} \Delta_5 &\leq c_1 \sum_{m=M_3+1}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} 2^{-j\delta - is'} 2^{in/t} [k^{1-\varepsilon} \cdot 2^{iz_3} \cdot 2^{jz_4}]^{-\frac{1}{2}} \\ &\leq c_2 k^{-\frac{1}{2}(1-\varepsilon)} \sum_{m=M_3+1}^{M_4} \sum_{\substack{i+j=m \\ i \geq j}} 2^{-j(\delta + \frac{z_4}{2})} 2^{-i(s' - \frac{n}{t} + \frac{z_3}{2})} \\ &\leq c_3 k^{-\frac{1}{2}(1-\varepsilon)} \sum_{m=M_3+1}^{M_4} 2^{-m(s' - \frac{n}{t} + \frac{z_3}{2})} \\ &\leq c_4 2^{-M_4 s'} = c_4 k^{-\frac{ts'}{2n}}. \end{aligned}$$

Summarizing all the estimates of the six parts (in fact,  $\Delta_2, \Delta_3, \Delta_5$  and  $\Delta_6$ ) in each case of (ii)-(iv), we obtain the upper bounds in these cases, respectively, as required. We wish to mention that, in case (ii), if  $\delta > s'$  and  $\delta - s' < \frac{2d-n}{t}$ , then the inequality  $2d > n$  is valid, and similarly in case (iv), the relation  $2d < n$  holds.

**Step 5.** The lower estimate of  $a_k(\text{id}, B_1, B_2)$ . Consider the following diagram

$$\begin{array}{ccc} \ell_{p_1}^{M_{j,i}} & \xrightarrow{S_{j,i}} & \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)) \\ \downarrow \text{id}_1 & & \downarrow \text{id} \\ \ell_{p_2}^{M_{j,i}} & \xleftarrow{T_{j,i}} & \ell_{q_2}(\ell_{p_2}) \end{array} \quad (3.30)$$

Here,

$$\begin{aligned} (S_{j,i}\eta)_{u,v} &:= \begin{cases} \eta_{\varphi(u,v)} & \text{if } (u,v) \in I_{j,i}, \\ 0 & \text{otherwise,} \end{cases} \\ (T_{j,i}\lambda)_{\varphi(u,v)} &:= \lambda_{u,v}, \quad (u,v) \in I_{j,i}, \end{aligned}$$

and  $\varphi$  denotes a bijection of  $I_{j,i}$  onto  $\{1, \dots, M_{j,i}\}$ ,  $j \in \mathbb{N}_0$ ,  $i \in \mathbb{N}_0$ ; cf. (3.14) and (3.15). Observe that

$$\begin{aligned} S_{j,i} &\in \mathcal{L} \left( \ell_{p_1}^{M_{j,i}}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(\alpha)) \right) \quad \text{and} \quad \|S_{j,i}\| = 2^{j\delta + is'}, \\ T_{j,i} &\in \mathcal{L} \left( \ell_{q_2}(\ell_{p_2}), \ell_{p_2}^{M_{j,i}} \right) \quad \text{and} \quad \|T_{j,i}\| = 1. \end{aligned}$$



Hence we obtain

$$a_k(\text{id}_1) \leq \|S_{j,i}\| \|T_{j,i}\| a_k(\text{id}). \quad (3.31)$$

In the following four points, we first assume  $p_2 \neq \infty$  if  $0 < p_1 \leq 1$ .

(a) Let  $\frac{1}{t} < \frac{\delta}{d} \leq \frac{s'}{n}$ . We consider the case  $i = 0, j \geq \frac{2}{d}$ . Lemma 3.3 implies that  $N_{j,0} \sim 2^{dj}$ . In view of (3.19), we observe that either  $M_{j,1}$  or  $M_{j,2}$  is no smaller than  $N_{j,0}/2$ . So by (3.18), we may assume that  $N := M_{j,1} = |I_{j,1}| \sim 2^{jd}$ . Moreover,

$$\|S_{j,1}\| \leq C2^{j\delta} \quad \text{and} \quad \|T_{j,1}\| = 1.$$

Put  $m = \lceil \frac{N}{4} \rceil \sim 2^{jd-2}$ . And for sufficiently large  $N$  we have  $m \geq N^{2/t}$  since  $t > 2$ . Consequently, we observe by Lemma 3.5 that

$$a_m(\text{id}_1, \ell_{p_1}^N, \ell_{p_2}^N) \sim N^{\frac{1}{t}} m^{-\frac{1}{2}} \sim 2^{(jd-2)(\frac{1}{t}-\frac{1}{2})}.$$

Using (3.31), we obtain

$$a_{2^{jd-2}}(\text{id}) \geq C_1 2^{-j\delta} 2^{(jd-2)(\frac{1}{t}-\frac{1}{2})} \geq C_2 2^{(jd-2)(\frac{1}{t}-\frac{1}{2}-\frac{\delta}{d})}.$$

Then the monotonicity of the approximation numbers implies that for any  $k \in \mathbb{N}$

$$a_k(\text{id}) \geq C_3 k^{-(\frac{\delta}{d}+\frac{1}{2}-\frac{1}{t})}. \quad (3.32)$$

(b) Let  $\frac{1}{t} < \frac{s'}{n} \leq \frac{\delta}{d}$ . We consider the case  $j = 0, j \geq \frac{2}{d}$ . Then, in a similar manner as above, we may assume that  $N := M_{0,i+1} = |I_{0,i+1}| \sim 2^{in}$ . Moreover,

$$\|S_{0,i+1}\| \leq C2^{is'} \quad \text{and} \quad \|T_{0,i+1}\| = 1.$$

Also put  $m = \lceil \frac{N}{4} \rceil \sim 2^{ni-2}$ . Hence we have similarly for any  $k \in \mathbb{N}$

$$a_k(\text{id}) \geq Ck^{-(\frac{s'}{n}+\frac{1}{2}-\frac{1}{t})}. \quad (3.33)$$

(c) Let  $\frac{\delta}{d} \leq \frac{1}{t}$  and  $\frac{\delta}{d} \leq \frac{s'}{n}$ . We select the same  $N, S$ , and  $T$  as in point (a) and take  $m = \lceil N^{\frac{2}{t}} \rceil \leq \frac{N}{4}$  for sufficiently large  $N$ . Then  $N^{\frac{1}{t}} m^{-\frac{1}{2}} \sim 1$ . Hence by Lemma 3.5 and (3.31) we obtain

$$a_m(\text{id}) \geq C2^{-j\delta} = C2^{-jd\frac{2}{t}\frac{\delta}{2d}},$$

and then for any  $k \in \mathbb{N}$

$$a_k(\text{id}) \geq Ck^{-\frac{t\delta}{2d}}. \quad (3.34)$$

(d) Let  $\frac{s'}{n} \leq \frac{1}{t}$  and  $\frac{s'}{n} \leq \frac{\delta}{d}$ . We select the same  $N, S$ , and  $T$  as in point (b) and take  $m = \lceil N^{\frac{2}{t}} \rceil$  in the same way as in point (c). Then analogously

$$a_m(\text{id}) \geq C2^{-is'} = C2^{-ni\frac{2}{t}\frac{s'}{2n}},$$

and in consequence, for any  $k \in \mathbb{N}$

$$a_k(\text{id}) \geq Ck^{-\frac{ts'}{2n}}. \quad (3.35)$$

If  $0 < p_1 \leq 1$  and  $p_2 = \infty$ , we have  $t = \infty$  and consider two cases,  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $0 < \frac{s'}{n} < \frac{\delta}{d}$ . And we choose  $m = \lceil \frac{N}{2} \rceil$  where  $N$  is taken in the same way as in point (a) or (b), respectively, now using (3.8) instead of Lemma 3.5.

Finally, we mention that in case (ii), the condition  $\delta > s'$  implies the inequality  $\frac{s'}{n} < \frac{\delta}{d}$ .

The proof of the proposition is now complete.  $\square$

**Remark 3.10.** *In the situation considered in Proposition 3.9, how do the approximation numbers behave, if  $s' < n/t$  and,  $\delta = s'$  or  $\delta - s' = \frac{2d-n}{t}$ ? The lower bound given in (3.34) may be the exact asymptotic estimate in some special cases.*

**Proposition 3.11.** *Suppose  $0 < p_1 \leq p_2 \leq 2$  or  $2 \leq p_1 \leq p_2 \leq \infty$ . Then*

$$a_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\varkappa}, \quad (3.36)$$

with

$$\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right). \quad (3.37)$$

*Proof.* The upper bound can be proved in the same way as in the proof of Theorem 6 in [17] using  $1/s$  instead of  $1/r - 1/p$ , where the latter notations follow from [17]. In fact, in the estimate from above, in view of Lemma 3.5 (i), we obtain that for any  $s > 0$ ,

$$L_{s,\infty}^{(a)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \leq C \begin{cases} 2^{(in+d(j-i))/s}, & 0 \leq i < j, \\ 2^{in/s}, & 0 \leq j \leq i. \end{cases} \quad (3.38)$$

In the estimate from below we only need to consider two cases. If  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  we take the same  $N$ ,  $S$  and  $T$  as in point (a) of Step 5 of the last proof. If  $0 < \frac{s'}{n} < \frac{\delta}{d}$  we choose the same  $N$ ,  $S$  and  $T$  as in point (b) therein.  $\square$

**Proposition 3.12.** *Suppose  $\tilde{p} < p_2 < p_1 \leq \infty$  where  $\frac{1}{\tilde{p}} = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1}$ . Then*

$$a_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\varkappa}, \quad (3.39)$$

with

$$\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2}. \quad (3.40)$$

*Proof.* The proof of the upper bound may be again finished as in the proof of Theorem 6 in [17] with  $1/r - 1/p$  replaced by  $1/s - 1/p_1 + 1/p_2$ , where the former notations follow from [17]. Indeed, in terms of Lemma 3.7, we observe that for any  $s > 0$ ,

$$L_{s,\infty}^{(a)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \leq C \begin{cases} 2^{(in+d(j-i))(\frac{1}{s} - \frac{1}{p_1} + \frac{1}{p_2})}, & 0 \leq i < j, \\ 2^{in(\frac{1}{s} - \frac{1}{p_1} + \frac{1}{p_2})}, & 0 \leq j \leq i. \end{cases} \quad (3.41)$$

In the estimate from below, once more we follow the proof of Step 5 of Proposition 3.9. In order to guarantee the compactness of the embeddings, here we only need to consider two cases,  $\frac{1}{p^*} < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $\frac{1}{p^*} < \frac{s'}{n} < \frac{\delta}{d}$ . And we choose  $m = \lceil \frac{N}{2} \rceil$  where  $N$  is taken in the same way as in point (a) or (b), respectively, now using Lemma 3.7 instead of Lemma 3.5.  $\square$

### 3.3 Kolmogorov numbers of sequence spaces

Now, we turn our attention to Kolmogorov numbers. To begin with, we shall collect some information on estimates for the Euclidean ball. Lemma 3.13 follows trivially from results of Gluskin [6] and Edmunds and Triebel [3].

**Lemma 3.13.** *Let  $N \in \mathbb{N}$ .*

(i) *If  $1 \leq p_1 \leq p_2 \leq 2$  and  $k \leq \frac{N}{4}$ , then*

$$d_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim 1.$$

(ii) *If  $1 \leq p_1 < 2 < p_2 < \infty$  and  $k \leq \frac{N}{4}$ , then*

$$d_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim \min\{1, N^{\frac{1}{p_2}} k^{-\frac{1}{2}}\}.$$

(iii) *If  $2 < p_1 = p_2 \leq \infty$  and  $k \leq N$ , then*

$$d_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim 1.$$

(ii) *If  $2 \leq p_1 < p_2 < \infty$  and  $k \leq N$ , then*

$$d_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim \xi^\theta,$$

$$\text{where } \xi = \min\{1, N^{\frac{1}{p_2}} k^{-\frac{1}{2}}\}, \theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2}.$$

The following lemma are a simply corollary of Lemma 3.13. And the proof can be finished in the same manner as in the proof of Lemma 10 in Skrzypczak [24].

**Lemma 3.14.** *Suppose  $1 \leq p_1 < 2 < p_2 < \infty$ . Then there is a positive constant  $C$  independent of  $N$  and  $k$  such that*

$$d_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \leq C \begin{cases} 1 & \text{if } k \leq N^{\frac{2}{p_2}}, \\ N^{\frac{1}{p_2}} k^{-\frac{1}{2}} & \text{if } N^{\frac{2}{p_2}} < k \leq N, \\ 0 & \text{if } k > N. \end{cases} \quad (3.42)$$

Now, we go on to make preparations for the estimates of Kolmogorov numbers of related embeddings for the quasi-Banach case with  $0 < p_1 < 1$  or  $0 < p_2 < 1$ , and for several cases left over when  $p_2 = \infty$ . The following result, Lemma 3.15, is due to Kashin [9], Garnaev and Gluskin [5] and Vybíral [29], cf. also [18].

**Lemma 3.15.** *Let  $N \in \mathbb{N}$ .*

(i) *If  $1 \leq p < 2$  and  $k \leq \frac{N}{4}$ , then*

$$k^{-1/2} \preceq d_k(\text{id}, \ell_p^N, \ell_\infty^N) \preceq k^{-1/2} \left( \log \left( \frac{eN}{k} \right) \right)^{3/2}.$$

(ii) If  $2 \leq p < \infty$  and  $k \leq N$ , then

$$\frac{1}{4} \min \left\{ 1, \left( c_1 \frac{\log(1 + \frac{N}{k-1})}{k-1} \right)^{1/p} \right\} \leq d_k(\text{id}, \ell_p^N, \ell_\infty^N) \leq \min \left\{ 1, \left( c_2 \frac{\log(1 + \frac{N}{k-1})}{k-1} \right)^{1/p} \right\}$$

are valid for certain absolute constants  $c_1 > 0$  and  $c_2 > 0$ .

(iii) If  $0 < p_1 < 1$ ,  $p_1 < p_2 \leq \infty$  and  $k \leq N$ , then

$$d_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) = d_k(\text{id}, \ell_{\min(1, p_2)}^N, \ell_{p_2}^N).$$

The following lemma is a simple corollary of Lemma 3.15. And the proof is similar to that of Lemma 10 in [24].

**Lemma 3.16.** *Let  $1 \leq p_1 < 2$  and  $N = 1, 2, 3, \dots$ . Then there is a positive constant  $C$  independent of  $N$  and  $k$  such that*

$$d_k(\text{id}, \ell_{p_1}^N, \ell_\infty^N) \leq C \begin{cases} k^{-1/2} \left( \log \left( \frac{4eN}{k} \right) \right)^{3/2} & \text{if } 0 < k \leq N, \\ 0 & \text{if } k > N. \end{cases} \quad (3.43)$$

The following estimate is due to Vybíral [29].

**Lemma 3.17.** *If  $0 < p_2 \leq p_1 \leq \infty$ , then there is a constant  $c$ ,  $0 < c \leq 1$ , such that*

$$d_{[ck]+1}(\text{id}, \ell_{p_1}^{2k}, \ell_{p_2}^{2k}) \succeq n^{1/p_2 - 1/p_1}, \quad k \in \mathbb{N}.$$

Now we are ready to deal with the Kolmogorov numbers of embeddings of related sequence spaces in the quasi-Banach setting,  $0 < p, q \leq \infty$ .

**Proposition 3.18.** *Suppose  $0 < p_1 < 2 < p_2 \leq \infty$ .*

(i) *If  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{p_2}$ , then*

$$d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\min(\frac{\delta}{d}, \frac{s'}{n}) + \frac{1}{p_2} - \frac{1}{2}}. \quad (3.44)$$

(ii) *If  $\delta > s'$  and either  $\begin{cases} \delta < \frac{d}{p_2}, \\ \delta - s' < \frac{2d-n}{p_2}, \end{cases}$  or  $\begin{cases} \delta + s' < \frac{n}{p_2}, \\ \delta - s' > \frac{2d-n}{p_2}, \end{cases}$  then*

$$d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\frac{s'}{n} \cdot \frac{p_2}{2}}. \quad (3.45)$$

(iii) *If  $\delta < \frac{d}{p_2}$ ,  $\delta < s' < \frac{n}{p_2}$  and  $\delta - s' < \frac{2d-n}{p_2}$ , then*

$$k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}. \quad (3.46)$$

(iv) If  $\delta < s'$ ,  $\delta + s' < \frac{n}{p_2}$  and  $\delta - s' > \frac{2d-n}{p_2}$ , then

$$k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{p_2}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}. \quad (3.47)$$

*Proof.* We consider three cases according to the distributions of  $p_1$  and  $p_2$ . Note that points (ii)-(iv) vanish when  $p_2 = \infty$ .

Case 1. Assume  $1 \leq p_1 < 2 < p_2 < \infty$ . The proof of the proposition can be finished in the same manner as in the proof of Proposition 3.9, with Lemma 3.5 and (3.9) replaced by Lemma 3.13 and (3.42), respectively. The only change is that  $t = \min(p'_1, p_2)$  is replaced by  $p_2$  in this case.

Case 2. Assume  $1 \leq p_1 < 2$  and  $p_2 = \infty$ . We proceed as above, now using Lemma 3.15 (i) and (3.43) instead of Lemma 3.13 and (3.42), respectively. Related computations of ideal quasi-norms herein are similar to the counterpart of entropy numbers, cf. [3, 16]. Indeed,

$$L_{s,\infty}^{(d)}(\text{id}, \ell_{p_1}^N, \ell_{\infty}^N) \leq CN^{\frac{1}{s}-\frac{1}{2}}, \quad \frac{1}{s} > \frac{1}{2}, \quad N \in \mathbb{N}.$$

Moreover, in the estimate of lower bounds, because of  $p_2 = \infty$ , we only need to consider two cases,  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $0 < \frac{s'}{n} < \frac{\delta}{d}$ , in the same way as in point (a) or (b) of Step 5 in the proof of Proposition 3.9, respectively, and take  $m = \lceil \frac{N}{4} \rceil$  in both cases based on Lemma 3.15 (i).

Case 3. Assume  $0 < p_1 < 1$  and  $2 < p_2 \leq \infty$ . We first transform the problem of this case to the above two cases (i.e., Case 1 for  $p_2 < \infty$ , Case 2 for  $p_2 = \infty$ ), by virtue of Lemma 3.15 (iii), and follow trivially them respectively. Note that the exact upper estimate here may also be provided by the corresponding statement about approximation numbers, cf. Proposition 3.9 and (2.3). □

**Proposition 3.19.** Suppose  $2 \leq p_1 < p_2 \leq \infty$ . We set  $\theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2}$ .

(i) If  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{\theta}{p_2}$ , then

$$d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\min(\frac{\delta}{d}, \frac{s'}{n}) - \frac{1}{p_1} + \frac{1}{p_2}}. \quad (3.48)$$

(ii) If  $\delta > s'$  and either  $\begin{cases} \delta < \frac{d}{p_2}\theta, \\ \delta - s' < \frac{2d-n}{p_2}\theta, \end{cases}$  or  $\begin{cases} \delta + s' < \frac{n}{p_2}\theta, \\ \delta - s' > \frac{2d-n}{p_2}\theta, \end{cases}$  then

$$d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\frac{s'}{n} \cdot \frac{p_2}{2}}. \quad (3.49)$$

(iii) If  $\delta < \frac{d}{p_2}\theta$ ,  $\delta < s' < \frac{n}{p_2}\theta$  and  $\delta - s' < \frac{2d-n}{p_2}\theta$ , then

$$k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}. \quad (3.50)$$

(iv) If  $\delta < s'$ ,  $\delta + s' < \frac{n}{p_2}\theta$  and  $\delta - s' > \frac{2d-n}{p_2}\theta$ , then

$$k^{-\frac{p_2}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq d_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{p_2}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}. \quad (3.51)$$

*Proof.* We consider two cases as follows. Note that points (ii)-(iv) vanish when  $p_2 = \infty$ .

Case 1. Assume  $2 \leq p_1 < p_2 < \infty$ . We only sketch the proof since, once more, we can use the similar reasoning. To shorten notations we shall put  $\tau = \frac{p_2}{\theta}$ ,  $h = \frac{2}{\theta}$  and  $\frac{1}{s} = \frac{1}{\gamma} + \frac{1}{h}$  for any  $s > 0$ . These simple transformations lead us to follow trivially from the proof of Proposition 3.9. Please note that in the upper estimate, by Lemma 3.13,

$$L_{h,\infty}^{(d)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_h}^{M_{j,i}}) \leq C \begin{cases} 2^{(in+d(j-i))/\tau}, & 0 \leq i < j, \\ 2^{\frac{in}{\tau}}, & 0 \leq j \leq i, \end{cases} \quad (3.52)$$

$$L_{s,\infty}^{(d)}(\text{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \leq C \begin{cases} 2^{(in+d(j-i))(\frac{1}{\tau} + \frac{1}{\gamma})}, & 0 \leq i < j, \frac{1}{s} > \frac{1}{h}, \\ 2^{in(\frac{1}{\tau} + \frac{1}{\gamma})}, & 0 \leq j \leq i, \frac{1}{s} > \frac{1}{h}. \end{cases} \quad (3.53)$$

Similarly, with respect to the estimate from below,  $t = \min(p'_1, p_2)$  is replaced by  $\tau = \frac{p_2}{\theta}$  in related places. One can consult our previous paper [30] for further details.

Case 2. Assume  $2 \leq p_1 < p_2 = \infty$ . We proceed as above, now using Lemma 3.15 (ii) instead. Again, computations of corresponding operator ideal quasi-norms start up as below,

$$L_{s,\infty}^{(d)}(\text{id}, \ell_{p_1}^N, \ell_{\infty}^N) \leq CN^{\frac{1}{s} - \frac{1}{p_1}}, \quad \frac{1}{s} > \frac{1}{p_1}, \quad N \in \mathbb{N}.$$

In the estimate of lower bounds, we only need to consider two cases,  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $0 < \frac{s'}{n} < \frac{\delta}{d}$ , as in Case 2 of the last proof, and take  $m = N$  in both cases.  $\square$

**Proposition 3.20.** *Suppose  $0 < p_1 \leq p_2 \leq 2$  or  $2 < p_1 = p_2 \leq \infty$ . Then*

$$d_k\left(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})\right) \sim k^{-\varkappa}, \quad (3.54)$$

with

$$\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right). \quad (3.55)$$

*Proof.* The upper estimate is provided by the corresponding statement about approximation numbers, cf. Proposition 3.11 and (2.3).

In the lower estimate, once more we follow the proof of Step 5 of Proposition 3.9. If  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  we take the same  $N$ ,  $S$  and  $T$  as in point (a). If  $0 < \frac{s'}{n} < \frac{\delta}{d}$  we take  $N$ ,  $S$  and  $T$  the same as in point (b). Moreover, in each of these two cases we choose  $m = \lfloor \frac{N}{4} \rfloor$  (if  $p_2 \geq 1$ ) or  $m = \lfloor \frac{c}{2} \cdot N \rfloor$  (if  $p_1 \leq p_2 < 1$ ) where  $c$  is the constant from Lemma 3.17, and we use Lemma 3.13 (if  $p_1 \geq 1$ ) or, Lemma 3.15 (iii) and Lemma 3.17 (if  $p_1 \leq p_2 < 1$ ) or, Lemma 3.13 and Lemma 3.15 (iii) (if  $p_1 < 1 \leq p_2$ ), instead of Lemma 3.5.  $\square$

**Proposition 3.21.** *Suppose  $0 < \tilde{p} < p_2 < p_1 \leq \infty$  where  $\frac{1}{\tilde{p}} = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1}$ . Then*

$$d_k\left(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})\right) \sim k^{-\varkappa}, \quad (3.56)$$

with

$$\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2}. \quad (3.57)$$

*Proof.* Once again the estimate from above is provided by the corresponding statement about approximation numbers, cf. Proposition 3.12 and (2.3).

Again, in the estimate from below we follow the proof of Step 5 of Proposition 3.9. We only need to consider two cases,  $\frac{1}{p^*} < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $\frac{1}{p^*} < \frac{s'}{n} < \frac{\delta}{d}$ , as in the proof of Proposition 3.12. And in each case we choose  $m = \lceil \frac{c}{2} \cdot N \rceil$ , where  $c$  is the constant from Lemma 3.17.  $\square$

### 3.4 Gelfand numbers of sequence spaces

In this subsection we deal with Gelfand numbers. First, we collect some necessary information on the behaviour of  $c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N)$ , cf. [4, 6, 21, 29], (2.1) and (2.2).

**Lemma 3.22.** *Let  $N \in \mathbb{N}$ .*

(i) *If  $2 \leq p_1 \leq p_2 \leq \infty$  and  $k \leq \frac{N}{4}$  then*

$$c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim 1.$$

(ii) *If  $1 < p_1 < 2 < p_2 \leq \infty$  and  $k \leq \frac{N}{4}$  then*

$$c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim \min\{1, N^{\frac{1}{p_1}} k^{-\frac{1}{2}}\}.$$

(iii) *If  $1 \leq p_1 = p_2 < 2$  and  $k \leq N$ , then*

$$c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim 1.$$

(iv) *If  $1 < p_1 < p_2 \leq 2$  and  $k \leq N$ , then*

$$c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \sim \xi^{\theta_1},$$

$$\text{where } \xi = \min\{1, N^{\frac{1}{p_1}} k^{-\frac{1}{2}}\}, \theta_1 = \frac{1/p_2' - 1/p_1'}{1/2 - 1/p_1'}.$$

The proof of this lemma follows by (2.1), (2.2) and Lemma 3.13.

The following result is due to Foucart et al.[4]. Note that the definition of Gelfand widths is used in [4]. Here we refer to it in our words.

**Lemma 3.23.** *Let  $1 \leq k \leq N < \infty$ .*

(i) *If  $0 < p_1 \leq 1$  and  $2 < p_2 \leq \infty$  then there exist constants  $C_1, C_2 > 0$  depending only on  $p_1$  and  $p_2$  such that*

$$C_1 \min\left\{1, \frac{\ln\left(\frac{N}{k-1}\right) + 1}{k-1}\right\}^{1/p_1 - 1/p_2} \leq c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \leq C_2 \min\left\{1, \frac{\ln\left(\frac{N}{k-1}\right) + 1}{k-1}\right\}^{1/p_1 - 1/2}.$$

(ii) *If  $0 < p_1 \leq 1$  and  $p_1 < p_2 \leq 2$  then there exist constants  $C_1, C_2 > 0$  depending only on  $p_1$  and  $p_2$  such that*

$$C_1 \min\left\{1, \frac{\ln\left(\frac{N}{k-1}\right) + 1}{k-1}\right\}^{1/p_1 - 1/p_2} \leq c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \leq C_2 \min\left\{1, \frac{\ln\left(\frac{N}{k-1}\right) + 1}{k-1}\right\}^{1/p_1 - 1/p_2}.$$

**Remark 3.24.** For the upper bounds, there is another result given by Vybíral [29], cf. Lemma 4.11, with a slight difference between them on the log-factors. But they are equivalent for our estimates of related upper bounds considered in Theorem 2.7.

**Lemma 3.25.** Let  $k \in \mathbb{N}$ .

(i) If  $0 < p_1 \leq 1$  and  $2 \leq p_2 \leq \infty$  then

$$c_k(\text{id}, \ell_{p_1}^{2k}, \ell_{p_2}^{2k}) \succeq k^{1/2-1/p_1}. \quad (3.58)$$

(ii) If  $0 < p_1 \leq 1$  and  $p_1 < p_2 \leq 2$  then

$$c_k(\text{id}, \ell_{p_1}^{2k}, \ell_{p_2}^{2k}) \succeq k^{1/p_2-1/p_1}. \quad (3.59)$$

The proof of this lemma follows literally [29], p. 567, by the multiplicativity of Gelfand numbers. In fact, The point (ii) of Lemma 3.23 may also imply point (ii) of Lemma 3.25.

**Lemma 3.26.** If  $1 \leq k \leq N < \infty$  and  $0 < p_2 \leq p_1 \leq \infty$ , then

$$c_k(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) = (N - k + 1)^{1/p_2-1/p_1}.$$

The proof of this lemma follows literally [21], Section 11.11.4, see also [23]. Indeed the original proof is used only for the Banach space case  $1 \leq p_2 \leq p_1 \leq \infty$ . However, the same proof works also in the quasi-Banach case  $0 < p_2 \leq p_1 \leq \infty$ .

Now we show some asymptotic estimates of Gelfand numbers of embeddings between related sequence spaces in the quasi-Banach setting,  $0 < p, q \leq \infty$ .

**Proposition 3.27.** Suppose  $0 < p_1 < 2 < p_2 \leq \infty$ .

(i) If  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{p_1'}$ , then

$$c_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\min(\frac{\delta}{d}, \frac{s'}{n}) - \frac{1}{p_1} + \frac{1}{2}}. \quad (3.60)$$

(ii) If  $\delta > s'$  and either  $\begin{cases} \delta < \frac{d}{p_1'}, \\ \delta - s' < \frac{2d-n}{p_1'} \end{cases}$  or  $\begin{cases} \delta + s' < \frac{n}{p_1'}, \\ \delta - s' > \frac{2d-n}{p_1'} \end{cases}$ , then

$$c_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \sim k^{-\frac{s'}{n} \cdot \frac{p_1'}{2}}. \quad (3.61)$$

(iii) If  $\delta < \frac{d}{p_1'}$ ,  $\delta < s' < \frac{n}{p_1'}$  and  $\delta - s' < \frac{2d-n}{p_1'}$ , then

$$k^{-\frac{p_1'}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})) \preceq k^{-\frac{p_1'}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}. \quad (3.62)$$



(iv) If  $\delta < s'$ ,  $\delta + s' < \frac{n}{p_1}$  and  $\delta - s' > \frac{2d-n}{p_1}$ , then

$$k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \left( \text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2}) \right) \preceq k^{-\frac{p'_1}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}. \quad (3.63)$$

*Proof.* We consider three cases from the standpoint of  $p_1$  and  $p_2$ . Note that points (ii)-(iv) vanish when  $0 < p_1 \leq 1$ .

Case 1. Assume  $1 < p_1 < 2 < p_2 \leq \infty$ . This is corresponding to Case 1 in the proof of Proposition 3.18. So we may deal with the proof exactly in the same manner in terms of Lemma 3.22 (ii). The changes begin with (3.20), where  $d_n$  is substituted by  $c_n$ . And the others go on trivially.

Case 2. Assume  $0 < p_1 \leq 1$  and  $2 < p_2 \leq \infty$ . We proceed as above. Related computations of ideal quasi-norms herein are finished by Lemma 3.23 (i). Moreover, in the estimate of lower bounds, because of  $0 < p_1 \leq 1$ , we consider two cases,  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $0 < \frac{s'}{n} < \frac{\delta}{d}$ , and take  $m = \lfloor \frac{N}{2} \rfloor$  in both cases based on (3.58).  $\square$

**Proposition 3.28.** Suppose  $0 < p_1 < p_2 \leq 2$ . We set  $\theta_1 = \frac{1/p'_2 - 1/p'_1}{1/2 - 1/p'_1}$ .

(i) If  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{\theta_1}{p'_1}$ , then

$$c_k \left( \text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2}) \right) \sim k^{-\min(\frac{\delta}{d}, \frac{s'}{n}) - \frac{1}{p_1} + \frac{1}{p_2}}. \quad (3.64)$$

(ii) If  $\delta > s'$  and either  $\begin{cases} \delta < \frac{d}{p'_1} \theta_1, \\ \delta - s' < \frac{2d-n}{p'_1} \theta_1, \end{cases}$  or  $\begin{cases} \delta + s' < \frac{n}{p'_1} \theta_1, \\ \delta - s' > \frac{2d-n}{p'_1} \theta_1, \end{cases}$  then

$$c_k \left( \text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2}) \right) \sim k^{-\frac{s'}{n} \cdot \frac{p'_1}{2}}. \quad (3.65)$$

(iii) If  $\delta < \frac{d}{p'_1} \theta_1$ ,  $\delta < s' < \frac{n}{p'_1} \theta_1$  and  $\delta - s' < \frac{2d-n}{p'_1} \theta_1$ , then

$$k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \left( \text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2}) \right) \preceq k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{\delta+s'}{2n})}. \quad (3.66)$$

(iv) If  $\delta < s'$ ,  $\delta + s' < \frac{n}{p'_1} \theta_1$  and  $\delta - s' > \frac{2d-n}{p'_1} \theta_1$ , then

$$k^{-\frac{p'_1}{2} \cdot \min(\frac{\delta}{d}, \frac{s'}{n})} \preceq c_k \left( \text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2}) \right) \preceq k^{-\frac{p'_1}{n} \cdot \min(\delta, \frac{\delta+s'}{4})}. \quad (3.67)$$

*Proof.* We consider two cases for  $p_1$  and  $p_2$ . Note that points (ii)-(iv) vanish when  $0 < p_1 \leq 1$ .

Case 1. Assume  $1 < p_1 < p_2 \leq 2$ . This is corresponding to Case 1 in the proof of Proposition 3.19. So we may deal with the proof exactly in the same manner in terms of Lemma 3.22 (iv) and the ideas from Case 1 of the last proof.

Case 2. Assume  $0 < p_1 \leq 1$  and  $p_1 < p_2 \leq 2$ . Once more we proceed exactly as in the proof of Theorem 6 in [17]. Here, Lemma 3.23 (ii) implies the computations of corresponding operator ideal quasi-norms. In the lower estimate, we consider two cases,  $0 < \frac{\delta}{d} \leq \frac{s'}{n}$  or  $0 < \frac{s'}{n} < \frac{\delta}{d}$ , as in Case 2 of the last proof, and take  $m = \lfloor \frac{N}{2} \rfloor$  in both cases based on Lemma 3.25 (ii) or (3.59).  $\square$

**Proposition 3.29.** *Suppose  $2 \leq p_1 \leq p_2 \leq \infty$  or  $0 < p_1 = p_2 < 2$ . Then*

$$c_k\left(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})\right) \sim k^{-\varkappa}, \quad (3.68)$$

with

$$\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right). \quad (3.69)$$

*Proof.* The proof of this proposition follows exactly as in the proof of Proposition 3.11 with Lemma 3.5 replaced by Lemma 3.22 and Lemma 3.26.  $\square$

**Proposition 3.30.** *Suppose  $0 < \tilde{p} < p_2 < p_1 \leq \infty$  where  $\frac{1}{\tilde{p}} = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1}$ . Then*

$$c_k\left(\text{id}, \ell_{q_1}(2^{j\delta} \ell_{p_1}(w)), \ell_{q_2}(\ell_{p_2})\right) \sim k^{-\varkappa}, \quad (3.70)$$

with

$$\varkappa = \min\left(\frac{\delta}{d}, \frac{s'}{n}\right) + \frac{1}{p_1} - \frac{1}{p_2}. \quad (3.71)$$

*Proof.* The proof of this proposition follows exactly as in the proof of Proposition 3.12 with Lemma 3.9 replaced by Lemma 3.26.  $\square$

**Remark 3.31.** *It is remarkable that all results in Section 3 are independent of  $q_1$  and  $q_2$ .*

## 4 Widths of embeddings of 2-microlocal Besov Spaces

Using basic properties of these  $s$ -numbers and Proposition 3.2, we have, for any  $s \in \{a, c, d\}$ ,

$$s_k\left(\text{id}, B_{p_1, q_1}^{s_1, s'_1}(\mathbb{R}^n, U), B_{p_2, q_2}^{s_2, s'_2}(\mathbb{R}^n, U)\right) \sim s_k\left(\text{id}, \ell_{q_1}(2^{j(s_1 - \frac{n}{p_1})} \ell_{p_1}(v_1)), \ell_{q_2}(2^{j(s_2 - \frac{n}{p_2})} \ell_{p_2}(v_2))\right),$$

with equivalence constants independent of  $k \in \mathbb{N}$ , cf. also (3.2). This leads us to transfer the results of Section 3 for sequence spaces back to function spaces. Theorem 2.3 follows from Proposition 3.9, Proposition 3.11 and Proposition 3.12. Theorem 2.5 follows from Propositions 3.18-3.21. Theorem 2.7 follows from Propositions 3.27-3.30.  $\square$

**Remark 4.1.** *If  $U = \{x_0\}$ , similar conclusions on the approximation, Gelfand and Kolmogorov numbers could be made for Corollary 8 in [17].*

**Remark 4.2.** *If  $U = \{0\}$ , the comparison between our main theorems and the known results on the approximation, Gelfand and Kolmogorov numbers of embeddings of Besov spaces with polynomial weights, cf. [24, 30], could also be easily made, as is shown for entropy numbers in Remark 4 of [17]. We do not go into detail.*

**Remark 4.3.** *Finally, we wish to mention some open questions. What is the asymptotic behavior of related  $n$ -widths for the other cases unsolved here (especially the case  $\frac{\delta}{d} \leq \frac{1}{\min(p'_1, p_2)} \leq \frac{s'}{n}$  with  $0 < p_1 < 2 < p_2 \leq \infty$  for the approximation numbers), under the equivalent condition of compactness,  $\min(\frac{\delta}{d}, \frac{s'}{n}) > \frac{1}{p^*}$ ? In some cases, the optimal order may even depend on the microscopic parameters  $q_1$  and  $q_2$ .*

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